

The Veronese Surface in $\text{PG}(5, 3)$ and Witt's $5-(12, 6, 1)$ Design*

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Abstract

A conic of the Veronese surface in $\text{PG}(5, 3)$ is a quadrangle. If one such quadrangle is replaced with its diagonal triangle, then one obtains a point model \mathcal{K} for Witt's $5-(12, 6, 1)$ design, the blocks being the hyperplane sections containing more than three (actually six) points of \mathcal{K} . As such a point model is projectively unique, the present construction yields an easy coordinate-free approach to some results obtained independently by H.S.M. Coxeter and G. Pellegrino, including a projective representation of the Mathieu group M_{12} in $\text{PG}(5, 3)$.

1 Introduction

Throughout this paper \mathbf{V} is a 3-dimensional vector space over $F := \text{GF}(3)$ and \mathbf{W} denotes the symmetric tensor product $\mathbf{V} \vee \mathbf{V}$. Occasionally, it will be convenient to use coordinates. We fix an ordered basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ of \mathbf{V} . It yields the ordered basis

$$(\mathbf{e}_0 \vee \mathbf{e}_0, 2\mathbf{e}_0 \vee \mathbf{e}_1, 2\mathbf{e}_0 \vee \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_1, 2\mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_2)$$

of \mathbf{W} . All coordinate vectors are understood with respect to one of these bases. The projective plane on \mathbf{V} is $\text{PG}(2, 3) = (\mathcal{P}(\mathbf{V}), \mathcal{L}(\mathbf{V}), \in)$, where

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$\mathcal{P}(\mathbf{V})$ and $\mathcal{L}(\mathbf{V})$ denote the sets of points and lines, respectively. Likewise we have $\text{PG}(5, 3) = (\mathcal{P}(\mathbf{W}), \mathcal{L}(\mathbf{W}), \in)$. The *Veronese mapping* is given by

$$\varphi : \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(\mathbf{W}), F\mathbf{a} \mapsto F(\mathbf{a} \vee \mathbf{a})$$

or, in terms of coordinates, by

$$F(x_0, x_1, x_2) \mapsto F(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2). \quad (1)$$

The set $\text{im } \varphi$ is the well-known *Veronese surface*. See, among others, [6, Chapter V], [8], [11, Chapter 25]. Recall three major properties of the Veronese mapping: Firstly, φ is injective. Secondly, the φ -image of each line l of $\text{PG}(2, 3)$ is a (non-degenerate) conic or, in other words, a planar quadrangle in $\text{PG}(5, 3)$. The plane of this conic meets $\text{im } \varphi$ in exactly four points. Each conic of $\text{im } \varphi$ arises in this way. Thirdly, the pre-image under φ of each hyperplane \mathcal{H} of $\text{PG}(5, 3)$ is a (possibly degenerate) quadric of $\text{PG}(2, 3)$. Each quadric of $\text{PG}(2, 3)$ arises in this way.

If we are given a quadrangle Γ in a projective plane of order 3, then its diagonal points form a triangle Δ , say. On the other hand, if Δ is a triangle in such a plane, then there are exactly four points which are not on any side of Δ . Those four points form a quadrangle, say Γ , which in turn has Δ as its diagonal triangle [9, 391–392]. This one-one correspondence between quadrangles and triangles in a projective plane of order three is the backbone of this paper.

There is also another interpretation of this correspondence: We may consider the quadrangle Γ as a conic. It will be called the *associated conic* of the triangle Δ . The internal points of the conic Γ comprise the triangle Δ . Moreover, Δ is a self polar triangle of Γ [9, Theorem 8.3.4.]. Finally, the sides of Δ are all the external lines of Γ .

2 Variations on $13 - 4 + 3 = 12$

In the sequel an arbitrarily chosen line l_∞ of $\text{PG}(2, 3)$ will be regarded as *line at infinity*. Its Veronese image $l_\infty^\varphi =: \Gamma_\infty$ is a planar quadrangle with diagonal triangle Δ_∞ , say. The plane spanned by Γ_∞ is denoted by \mathcal{E}_∞ .

The following Theorem describes the essential construction:

Theorem 1 *Write \mathcal{K} for that set of points in $\text{PG}(5, 3)$ which is obtained from the Veronese surface $\text{im } \varphi$ by replacing the planar quadrangle Γ_∞ , i.e. the φ -image of the line at infinity, with its diagonal triangle Δ_∞ . Then the following hold true:*

$$d_{\mathcal{H}} := \#(\mathcal{H} \cap \mathcal{K}) \in \{0, 3, 6\} \text{ for all hyperplanes } \mathcal{H} \text{ of } \text{PG}(5, 3). \quad (2)$$

$$\#\mathcal{K} = 12. \quad (3)$$

Proof. The pre-image of \mathcal{H} under φ is a quadric of $\text{PG}(2, 3)$, say \mathcal{Q} . There are four cases [9, 140].

1. $\mathcal{E}_\infty \subset \mathcal{H}$: Hence $d_{\mathcal{H}} = \#\mathcal{Q} - 4 + 3$. As $l_\infty \subset \mathcal{Q}$, we obtain that \mathcal{Q} is the repeated line l_∞ or a cross of lines. Thus $d_{\mathcal{H}} = 4 - 4 + 3 = 3$ or $d_{\mathcal{H}} = 7 - 4 + 3 = 6$.
2. $\mathcal{E}_\infty \cap \mathcal{H}$ is an external line of Γ_∞ : Hence $d_{\mathcal{H}} = \#\mathcal{Q} - 0 + 2$. As \mathcal{Q} is either a single affine point or a conic without points at infinity, we infer $d_{\mathcal{H}} = 1 - 0 + 2 = 3$ or $d_{\mathcal{H}} = 4 - 0 + 2 = 6$.
3. $\mathcal{E}_\infty \cap \mathcal{H}$ is a tangent of Γ_∞ : A tangent carries no internal points so that $d_{\mathcal{H}} = \#\mathcal{Q} - 1 + 0$. The quadric \mathcal{Q} is a repeated line l with $l \neq l_\infty$, or a cross of lines with double point at infinity, but each line other than l_∞ , or a conic touching l_∞ . Thus $d_{\mathcal{H}} = 4 - 1 + 0 = 3$ or $d_{\mathcal{H}} = 7 - 1 + 0 = 6$ or $d_{\mathcal{H}} = 4 - 1 + 0 = 3$.
4. $\mathcal{E}_\infty \cap \mathcal{H}$ is a bisecant of Γ_∞ : A bisecant carries exactly one internal point, whence $d_{\mathcal{H}} = \#\mathcal{Q} - 2 + 1$. Now \mathcal{Q} is a cross of lines with double point not at infinity, or a conic with two distinct points at infinity. Hence $d_{\mathcal{H}} = 7 - 2 + 1 = 6$ or $d_{\mathcal{H}} = 4 - 2 + 1 = 3$.

Finally, $\text{im } \varphi \cap \mathcal{E}_\infty = \Gamma_\infty$ implies $\#\mathcal{K} = 13 - 4 + 3 = 12$. \square

Remark 1 If l_∞ is chosen to be the line $x_0 = 0$, then Δ_∞ can easily be expressed in terms of coordinates as

$$\{F(0, 0, 0, 1, 0, 1), F(0, 0, 0, 2, 1, 1), F(0, 0, 0, 2, 2, 1)\}. \quad (4)$$

Thus, by virtue of (1) and (4), one may describe \mathcal{K} in terms of coordinates.

Before we are going to reverse the construction of Theorem 1, we prove the following

Lemma 1 *Let \mathcal{K} be a set of points in $\text{PG}(5, 3)$. Then (2) and (3) together are equivalent to the conjunction of the following three conditions:*

$$\text{Any 5-subset of } \mathcal{K} \text{ is independent.} \quad (5)$$

$$\#(\mathcal{H} \cap \mathcal{K}) \geq 5 \text{ implies } \#(\mathcal{H} \cap \mathcal{K}) = 6 \text{ for all hyperplanes } \mathcal{H} \text{ of } \text{PG}(5, 3) \quad (6)$$

$$\#\mathcal{K} \geq 7. \quad (7)$$

Proof. (2) and (3) \implies (5) and (6) and (7): Choose any 5-set $\mathcal{M} \subset \mathcal{K}$ and $P \in \mathcal{K} \setminus \mathcal{M}$. At first we are going to show that

$$\dim \operatorname{span}(\mathcal{M} \cup \{P\}) \geq 4; \quad (8)$$

here “dim” denotes the projective dimension. Assume to the contrary that $\dim \operatorname{span}(\mathcal{M} \cup \{P\}) < 4$. Then each hyperplane of $\operatorname{PG}(5, 3)$ passing through $\mathcal{M} \cup \{P\}$ meets \mathcal{K} in exactly six points, by (2). All those hyperplanes are covering \mathcal{K} , whence $\mathcal{K} = \mathcal{M} \cup \{P\}$, in contradiction to (3).

We infer from (8) that $\dim \operatorname{span} \mathcal{M} \geq 3$. This dimension cannot equal three, since then \mathcal{K} would only have nine points, namely the five points in \mathcal{M} plus one more point in each of the four hyperplanes through \mathcal{M} . Consequently, \mathcal{M} is independent. By (2) and (3), conditions (6) and (7) follow immediately.

(5) and (6) and (7) \implies (2) and (3): By our assumptions, \mathcal{K} contains a basis \mathcal{S} of $\operatorname{PG}(5, 3)$. Each of the six hyperplane faces of that basis contains exactly one more point of \mathcal{K} ; it is in general position with respect to the remaining five. Thus we have $\#\mathcal{K} \geq 12$. On the other hand choose four points in \mathcal{S} . Each of the four hyperplanes passing through them meets \mathcal{K} in at most six points. Hence $\#\mathcal{K} \leq 12$. Thus (3) holds true.

If we fix one 3-set $\Delta \subset \mathcal{K}$, then the number hyperplanes through Δ is 13, and the number of 2-sets in $\mathcal{K} \setminus \Delta$ is 36. By (5) and (6), the number of hyperplanes through Δ , meeting \mathcal{K} in exactly six points, is $36/3 = 12$. Hence there is a unique hyperplane \mathcal{H}_Δ , say, with

$$\Delta = \mathcal{K} \cap \mathcal{H}_\Delta. \quad (9)$$

Next fix one point $P \in \mathcal{K}$. There are 330 4-subsets of $\mathcal{K} \setminus \{P\}$. They give rise to the $330/5 = 66$ hyperplanes through P meeting \mathcal{K} in six points. Likewise one finds $\binom{11}{2} = 55$ triangles in \mathcal{K} containing P . Each of those triangles yields exactly one hyperplane through P meeting \mathcal{K} in three points only. There are, however, only $121 = 66 + 55$ hyperplanes through P , whence (2) follows. \square

Theorem 1 can be reversed now as follows:

Theorem 2 *Let \mathcal{K} be a set of points in $\operatorname{PG}(5, 3)$ satisfying (2) and (3). Suppose that \mathcal{V} is obtained from \mathcal{K} by replacing one triangle $\Delta \subset \mathcal{K}$ with its associated conic Γ . Then \mathcal{V} is projectively equivalent to the Veronese surface $\operatorname{im} \varphi$.*

Proof. By Lemma 1, there is a triangle $\Delta \subset \mathcal{K}$. The plane spanned by Δ is denoted by \mathcal{E} . According to [11, Theorem 25.3.14] it is sufficient to verify

the following conditions:

$$c_{\mathcal{H}} := \#(\mathcal{H} \cap \mathcal{V}) \in \{1, 4, 7\} \text{ for all hyperplanes } \mathcal{H} \text{ of } \text{PG}(5, 3). \quad (10)$$

$$c_{\mathcal{H}_0} = 7 \text{ for some hyperplane } \mathcal{H}_0 \text{ of } \text{PG}(5, 3). \quad (11)$$

In order to establish (10) choose a hyperplane \mathcal{H} and put $d_{\mathcal{H}} := \#(\mathcal{H} \cap \mathcal{K})$. There are four cases.

1. $\mathcal{E} \subset \mathcal{H}$: By (2), $c_{\mathcal{H}} = d_{\mathcal{H}} - 3 + 4 \in \{1, 4, 7\}$.
2. $\mathcal{E} \cap \mathcal{H}$ is an external line of Γ : Thus $\#(\mathcal{H} \cap \Delta) = 2$ and $c_{\mathcal{H}} = d_{\mathcal{H}} - 2 + 0 \in \{1, 4\}$.
3. $\mathcal{E} \cap \mathcal{H}$ is a tangent of Γ : Thus $\#(\mathcal{H} \cap \Delta) = 0$ and $c_{\mathcal{H}} = d_{\mathcal{H}} - 0 + 1 \in \{1, 4, 7\}$.
4. $\mathcal{E} \cap \mathcal{H}$ is a bisecant of Γ : Thus $\#(\mathcal{H} \cap \Delta) = 1$ and $c_{\mathcal{H}} = d_{\mathcal{H}} - 1 + 2 \in \{4, 7\}$.

Two points in $\mathcal{K} \setminus \Delta$ together with Δ generate a hyperplane \mathcal{H}_0 meeting \mathcal{K} in six distinct points by (5). According to case 1, $c_{\mathcal{H}_0} = 7$. \square

All properties of the Veronese surface that are used in the following proof can be read off, e.g., from [11, Section 25.1].

Theorem 3 *Suppose that $\mathcal{K}, \mathcal{K}'$ are sets of points in $\text{PG}(5, 3)$ subject to (2) and (3). Choose five distinct points P_0, \dots, P_4 in \mathcal{K} and five distinct points P'_0, \dots, P'_4 in \mathcal{K}' . Then there is a unique collineation κ of $\text{PG}(5, 3)$ with $\mathcal{K}^\kappa = \mathcal{K}'$ and $P_i^\kappa = P'_i$ for $i = 0, \dots, 4$.*

Proof. Put $\Delta := \{P_0, P_1, P_2\}$. Define Γ and \mathcal{V} according to Theorem 2. Write \mathcal{C} for the set of all conics contained in \mathcal{V} . Then $(\mathcal{V}, \mathcal{C}, \in)$ is a projective plane of order 3. Moreover, the Veronese mapping φ yields a collineation of $\text{PG}(2, 3)$ onto that projective plane. There is a unique conic in \mathcal{V} joining P_3 with P_4 . It meets Γ in a single point, say G_3 . The line spanned by G_3 and P_i ($i = 0, 1, 2$) is a bisecant of Γ , as it contains the internal point P_i ; hence it meets the conic Γ residually in a point G_i , say. Thus $\Gamma = \{G_0, \dots, G_3\}$. The four points $\{P_3, P_4, G_0, G_1\}$ form a “quadrangle” of the projective plane $(\mathcal{V}, \mathcal{C}, \in)$, i.e. a set of four points no three of which are on a common conic $\subset \mathcal{V}$.

Repeat the previous construction with \mathcal{K}' to obtain Δ' etc. By Theorem 2, there exists a collineation μ of $\text{PG}(5, 3)$ with $\mathcal{V}^\mu = \mathcal{V}'$. Thus $\{P_3^\mu, P_4^\mu, G_0^\mu, G_1^\mu\}$ is a “quadrangle” of the projective plane $(\mathcal{V}', \mathcal{C}', \in)$. There is a projective collineation λ' of $(\mathcal{V}', \mathcal{C}', \in)$ with

$$P_3^\mu \mapsto P'_3, P_4^\mu \mapsto P'_4, G_0^\mu \mapsto G'_0, G_1^\mu \mapsto G'_1.$$

This λ' extends to a projective collineation λ of $\text{PG}(5, 3)$. The product $\kappa := \mu\lambda$ has the required properties, since $G_3^\kappa = G_3'$ implies $G_2^\kappa = G_2'$, so that also

$$P_i^\kappa = P_i' \text{ for } i = 0, 1, 2.$$

If $\bar{\kappa}$ is a collineation subject to the conditions of the theorem, then $\bar{\kappa}\kappa^{-1}$ restricts to a collineation of $(\mathcal{V}, \mathcal{C}, \in)$ fixing each point of a “quadrangle”. Now $\text{Aut GF}(3) = \{\text{id}\}$ forces $\bar{\kappa}\kappa^{-1}$ to fix \mathcal{V} pointwise, whence $\bar{\kappa} = \kappa$. \square

In the sequel let \mathcal{K} be the subset of $\text{PG}(5, 3)$ described in Theorem 1.

Remark 2 By Theorem 3, any set of points in $\text{PG}(5, 3)$ satisfying (2) and (3) is projectively equivalent to \mathcal{K} . We infer from Lemma 1 and Theorem 3 that the 12-sets of points discussed in [7] and [14] are essentially our \mathcal{K} . By [14, Teorema 4.3], conditions (3) and (5) characterize \mathcal{K} to within projective collineations. The set \mathcal{K} has a lot of fascinating geometric properties [7], [14], [16].

Remark 3 Define a *block* of \mathcal{K} as a hyperplane section of \mathcal{K} containing more than three points. If \mathcal{B} denotes the set of all such blocks, then the incidence structure $(\mathcal{K}, \mathcal{B}, \in)$ is *Witt’s 5-(12, 6, 1) design* W_{12} ; cf., e.g., [3, Chapter 4]. According to Lemma 1, Theorem 2, and Theorem 3, such a point model of W_{12} in $\text{PG}(5, 3)$ is projectively unique.

Remark 4 The automorphism group of W_{12} is the *Mathieu group* M_{12} , a sporadic simple group acting sharply 5-transitive on \mathcal{K} ; cf., e.g., [3, Chapter 4]. Each automorphism of $(\mathcal{K}, \mathcal{B}, \in)$ extends to a unique automorphic collineation of \mathcal{K} [7], [14]. Theorem 3 includes a short coordinate-free proof of that result.

Remark 5 The successive derivations of W_{12} are a 4-(11, 5, 1) design, a 3-(10, 4, 1) design (the *Möbius plane* over the field extension $\text{GF}(9)/\text{GF}(3)$), and a 2-(9, 3, 1) design (the *affine plane* over $\text{GF}(3)$). One may obtain point models for them by suitable projections of \mathcal{K} . Projection through a point of \mathcal{K} yields an 11-cap in a hyperplane of $\text{PG}(5, 3)$. See [10], [13], [14], [15]. If the centre of projection is a bisecant of \mathcal{K} , then one gets an *elliptic quadric* in a solid of $\text{PG}(5, 3)$. Finally, if the centre of projection is spanned by a triangle of \mathcal{K} , then an *affine subplane* of a projective plane of $\text{PG}(5, 3)$ arises. If the triangle is chosen to be Δ_∞ , then there exists an affinity of this affine plane onto $\mathcal{P}(\mathbf{V}) \setminus l_\infty$. This is immediately seen from (1) and (4) by projecting, e.g., onto the plane with equations $x_{11} = x_{12} = x_{22} = 0$.

Remark 6 Let $F^{\mathcal{P}(\mathbf{W})}$ be the F -vector space of all functions $\mathcal{P}(\mathbf{W}) \rightarrow F$. Given $\mathcal{M} \subset \mathcal{P}(\mathbf{W})$ denote by $\chi(\mathcal{M}) \in F^{\mathcal{P}(\mathbf{W})}$ its characteristic vector (function). With the notations of Theorem 1 we obtain

$$\chi(\text{im } \varphi) - \chi(\Gamma_\infty) + \chi(\Delta_\infty) = \chi(\mathcal{K}).$$

The characteristic vectors of the hyperplanes $\mathcal{H} \subset \mathcal{P}(\mathbf{W})$ are spanning a linear [364, 22, 121]-code [2, Theorem 5.7.1]. By (2), $\chi(\mathcal{K})$ is a word of weight 12 in the orthogonal (dual) code, where orthogonality is understood with respect to the standard dot product. According to (10), the Veronese variety yields a word of weight 13 which has dot product $1 \in F$ with each hyperplane. Thus, in terms of characteristic vectors, \mathcal{K} arises from the Veronese variety by adding a word of weight 7 which has dot product $2 \in F$ with each hyperplane.

Next let $\mathbf{w}_1, \dots, \mathbf{w}_{12} \in \mathbf{W}$ be vectors representing the points of \mathcal{K} . As f ranges over the dual vector space \mathbf{W}^* , the words $(\mathbf{w}_1^f, \dots, \mathbf{w}_{12}^f) \in F^{12}$ give the *extended ternary Golay code* \mathbf{G}_{12} . Cf. [1], where the dual point of view has been adopted. If we start instead with vectors $\mathbf{v}_1 \vee \mathbf{v}_1, \dots, \mathbf{v}_{13} \vee \mathbf{v}_{13}$ ($\mathbf{v}_i \in \mathbf{V}$) representing the points of the Veronese surface, then we obtain a ternary [13, 6, 6]-code \mathbf{C} , as follows from $\text{span im } \varphi = \mathcal{P}(\mathbf{W})$ and (10).

Given $f \in \mathbf{W}^*$ then $q : \mathbf{V} \rightarrow F$, $\mathbf{a} \mapsto (\mathbf{a} \vee \mathbf{a})^f$ is a quadratic form. The mapping $f \mapsto q$ is a linear bijection of \mathbf{W}^* onto the vector space of quadratic forms $\mathbf{V} \rightarrow F$. Thus, as q ranges over all quadratic forms on \mathbf{V} , the words $(\mathbf{v}_1^q, \dots, \mathbf{v}_{13}^q)$ too comprise the code \mathbf{C} .

In order to identify the code \mathbf{C} , let $\mathbf{C}(p)$ (p prime) be the linear code over $\text{GF}(p)$ which is spanned by the characteristic vectors of the lines of $\text{PG}(2, p)$. The dimension of $\mathbf{C}(p)$ is $(p^2 + p + 2)/2$, $\mathbf{C}(p)^\perp \subset \mathbf{C}(p)$, and $\mathbf{C}(p)^\perp$ has codimension 1 in $\mathbf{C}(p)$ [2, 49]. Moreover, $\mathbf{C}(p)^\perp$ coincides with two other codes arising from $\text{PG}(2, p)$: One is the code $\mathbf{E}(p)$ spanned by the differences of characteristic vectors of lines [2, Theorem 6.3.1], the other is the code $\mathbf{C}'(p)$ spanned by the characteristic vectors of the complements of lines, as follows easily from $\mathbf{C}'(p) \subset \mathbf{C}(p)^\perp$ and $\dim \mathbf{C}'(p) = \dim \mathbf{C}(p)^\perp$ [5, 366].

If a quadratic form $q : \mathbf{V} \rightarrow F$ is applied to four vectors \mathbf{v}_i which represent the points of a line, then one of the following (unordered) quadruples arises: $(0, 0, 0, 0)$, $\pm(1, 1, 1, 0)$, $(1, 2, 0, 0)$, $(1, 1, 2, 2)$. This is immediate from [9, Lemma 5.2.1]. Hence $\mathbf{C} \subset \mathbf{C}(3)^\perp$ and, by $\dim \mathbf{C} = \dim \mathbf{C}(3)^\perp$, the two codes turn out to be the same.

So, the self-dual extended ternary Golay code $\mathbf{G}_{12} = \mathbf{G}_{12}^\perp$ is closely related to a self-orthogonal code $\mathbf{C} \subset \mathbf{C}^\perp = \mathbf{C}(3)$ which belongs to an infinite family of codes obtained from $\text{PG}(2, p)$.

Remark 7 We aim at representing the points of Δ_∞ on the line l_∞ by making use of the Veronese mapping φ : Each bijection of l_∞ is a projectivity.

There are three *elliptic involutions* on l_∞ , each interchanging the points of l_∞ in pairs. Transformation under φ yields three elliptic involutions on the conic Γ_∞ . Each of them extends uniquely to a harmonic homology of the plane \mathcal{E}_∞ leaving Γ_∞ fixed, as a set [4, 2.4.4]. The centres of the three homologies are three distinct internal points of Γ_∞ , whence they comprise the set Δ_∞ . Thus the points of Δ_∞ are in one-one correspondence with the three elliptic involutions on l_∞ .

Now it is natural to ask for a description of W_{12} in terms of the nine points in $\mathcal{P}(\mathbf{V}) \setminus l_\infty$ and the three elliptic involutions on l_∞ . It turns out that one obtains Lüneburg's description [12, Chapter 7], although from a different point of view. A block is precisely one of the following:

1. An affine line plus all three elliptic involutions.
2. An ellipse together with those two elliptic involutions which are *not* the involution of conjugate points on l_∞ with respect to the ellipse.
3. A union of two distinct parallel affine lines.
4. A cross of affine lines together with that elliptic involution which interchanges the points at infinity of the two lines.

Cf. the proof of Theorem 1. Thus each block arises from an affine quadric and certain elliptic involutions which are affine invariants of the quadric. This observation was the starting point for the present paper.

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